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# VECTORS FOR BEGINNERS

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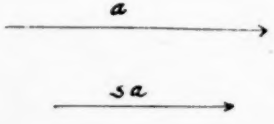
The vector idea is so common in the study of science and the vector method so easy and effectual in application that students in preparatory schools should acquire an understanding of the subject. Its importance is emphasized when we recall that displacement, velocity, acceleration, force, electric current, stresses, strains and many other physical quantities can be correctly represented by vectors.

A vector is a quantity that has magnitude and direction and can therefore be represented by a directed segment of a straight line. A scalar quantity lacks the quality of direction and ordinarily may be considered as of the nature of real number varying from $-\infty$ through zero to $+\infty$ like the numbers on a linear scale which proceed either negatively or positively from the zero point.




If we represent a vector **a** by a directed segment then $s\mathbf{a}$, $s$ being a scalar quantity is a vector parallel to **a** which is less than **a** if $0 < s < 1$ and greater than **a** if $s > 1$. If $s$ is negative, the vector $s\mathbf{a}$ is then directed in the opposite direction from that of **a**.

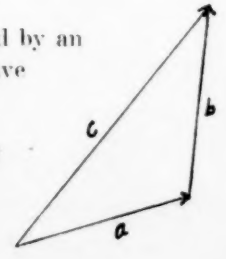
If $c$ is the third side of a triangle, the other sides being $a$ and $b$ directed as shown, $c$ is said to be the vector sum of $a$ and $b$ or

$$\mathbf{c} = \mathbf{a} + \mathbf{b}.$$

If we think of the vector **b** as being replaced by an equal and opposite vector $-\mathbf{b}$ we should have

$$\mathbf{c} + (-\mathbf{b}) = \mathbf{a} \text{ or } \mathbf{c} - \mathbf{b} = \mathbf{a},$$

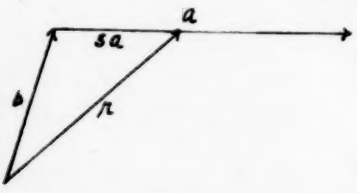
that is, we may transpose as in an ordinary algebraic equation or subtract vectors if we pay due regard to the direction of the vectors.



Again, if we have the vector equation

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{d}$$

one can easily verify that we have the same result whether we add **a** to the sum of **b** and **c** or **b** to the sum of **a** and **c** or **c** to the sum of **a** and **b**; that is the commutative and associative laws of algebraic addition hold.




In the equation

$$\mathbf{r} = \mathbf{b} + s\mathbf{a}$$

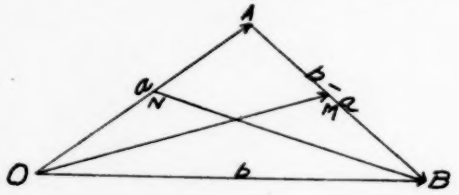
**r** is a vector from the origin of **b** to any point on **a** or **a** extended, depending upon the value of $s$.

It is evident that the sum of two non-parallel vectors cannot be zero unless each vector is zero, so that, if

$$(s' - s)\mathbf{a} + (t' - t)\mathbf{b} = 0$$

$s$, $s'$, $t$ and $t'$ being scalars we must have $s - s' = 0$ and $t - t' = 0$ or $s = s'$ and $t = t'$.

This is sufficient knowledge to begin making applications to problems in geometry. To prove that the medians of a triangle intersect in a point two-thirds the distance from each vertex to the opposite side we assume **a** and **b** as vectors for the sides $OA$ and $OB$ of the triangle. Then $\mathbf{b} - \mathbf{a}$ is the third side $\overrightarrow{AB}$, that is directed from $A$ to $B$. Then the median $\overrightarrow{OM}$ is




$$\overrightarrow{OA} + \overrightarrow{AM} = \mathbf{a} + \tfrac{1}{2}(\mathbf{b} - \mathbf{a}) = \tfrac{1}{2}(\mathbf{a} + \mathbf{b})$$

Hence for the vector from $O$ to any point on $OM$ we have

$$\mathbf{r} = \tfrac{1}{2}s(\mathbf{a} + \mathbf{b})$$

For the median $\overrightarrow{NB}$ we have $\overrightarrow{ON} + \overrightarrow{NB} = \overrightarrow{OB}$ or $\overrightarrow{NB} = \mathbf{b} - \tfrac{1}{2}\mathbf{a}$ and since $N$ is the origin of this vector we have with $O$ as origin for any point on it.

$$\mathbf{r} = \overrightarrow{ON} + t(\overrightarrow{NB}) = \tfrac{1}{2}\mathbf{a} + t(\mathbf{b} - \tfrac{1}{2}\mathbf{a})$$

where the vectors $\vec{OM}$ and $\vec{NB}$ intersect the $\mathbf{r}$'s are identical and therefore

$$\tfrac{1}{2}s(\mathbf{a}+\mathbf{b}) = \tfrac{1}{2}\mathbf{a} + t(\mathbf{b} - \tfrac{1}{2}\mathbf{a})$$

or

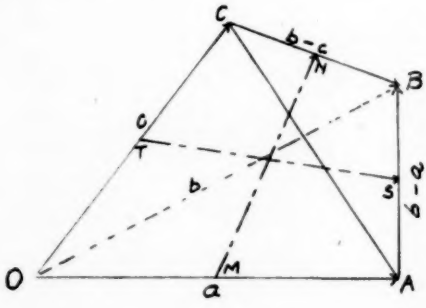
$$\tfrac{1}{2}(s+t-1)\,\mathbf{a} + (\tfrac{1}{2}s - t)\,\mathbf{b} = 0$$

Now, since the coefficients of $\mathbf{a}$ and $\mathbf{b}$ must vanish

$$s + t - 1 = 0 \text{ and } \tfrac{1}{2}s - t = 0$$

giving $t = \frac{1}{3}$ and $s = \frac{2}{3}$; that is the medians intersect two-thirds the distance from $O$ to $M$, and one-third the distance from $N$ to $B$.

Again, let us take the proposition from solid geometry that the lines from the middle points of opposite sides of a tetraedron meet and bisect each other.




In the tetraedron $OABC$ let $\vec{OA}$ be $\mathbf{a}$, $\vec{OB}$ be $\mathbf{b}$, $\vec{OC}$ be $\mathbf{c}$, then $\vec{AB} = \mathbf{b} - \mathbf{a}$, $\vec{CB} = \mathbf{b} - \mathbf{c}$ and $\vec{AC} = \mathbf{c} - \mathbf{a}$.

For the vector $\vec{MN}$ from the middle of $OA$ to the middle of $CB$ we have

$$\tfrac{1}{2}\mathbf{a} + \vec{MN} = \mathbf{c} + \tfrac{1}{2}(\mathbf{b} - \mathbf{c})$$

whence

$$\vec{MN} = \tfrac{1}{2}(\mathbf{b} + \mathbf{c} - \mathbf{a})$$

and for any point on $MN$ we have, $O$ being the origin

$$\mathbf{r} = \tfrac{1}{2}\mathbf{a} + \tfrac{1}{2}s(\mathbf{b} + \mathbf{c} - \mathbf{a})$$

For the vector $\vec{TS}$ from the middle of $OC$ to the middle of $AB$ we have

$$\tfrac{1}{2}\mathbf{c} + \vec{TS} = \mathbf{a} + \tfrac{1}{2}(\mathbf{b} - \mathbf{a})$$

giving

$$\vec{TS} = \tfrac{1}{2}(\mathbf{a} + \mathbf{b} - \mathbf{c})$$

so for any point on $TS$, $O$ being origin

$$\mathbf{r} = \tfrac{1}{2}\mathbf{c} + \tfrac{1}{2}t(\mathbf{a} + \mathbf{b} - \mathbf{c})$$

$s$ and $t$ being scalar quantities. If these intersect

$$\tfrac{1}{2}\mathbf{a} + \tfrac{1}{2}s(\mathbf{b} + \mathbf{c} - \mathbf{a}) = \tfrac{1}{2}\mathbf{c} + \tfrac{1}{2}t(\mathbf{a} + \mathbf{b} - \mathbf{c})$$

and it must be possible for $s$ and $t$ to satisfy the three equations:

$$1 - s - t = 0 \qquad s - 1 + t = 0 \qquad s - t = 0$$

which can be done by $s = \frac{1}{2}$ and $t = \frac{1}{2}$.

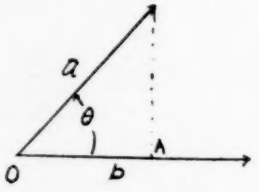
The two vectors therefore intersect and at their middle points. The ease and directness of this method will certainly appeal to any pupil who has the elements of a real student in him.

The absolute length of a vector $\mathbf{a}_o$, usually written $\mathbf{a}_o$, is its magnitude without regard to its direction. The absolute length of the vector 5 miles northwest is 5 miles.

The *dot* or *scalar* product of two vectors (written $a \cdot b$) is a scalar quantity equal to the product of the absolute values of one vector times the projection of the other upon it. Thus:

$$\mathbf{a} \cdot \mathbf{b} = (OA)\mathbf{b}_o = \mathbf{a}_o\mathbf{b}_o \cos\theta$$

where $\theta$ is the angle between $\mathbf{a}$ and $\mathbf{b}$.




If $\theta$ is zero we have, letting $\mathbf{b} = \mathbf{a}$,

$$\mathbf{a} \cdot \mathbf{a} = \mathbf{a}_o^2 = a^2$$

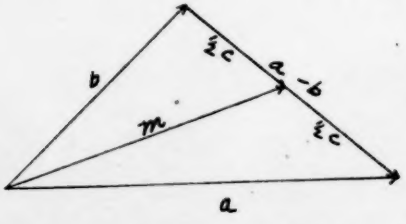
so we get the important relation

$$a = \mathbf{a}_o = \sqrt{\mathbf{a} \cdot \mathbf{a}}$$

or the absolute length of a vector is the square root of the dot product of the vector by itself. Dot products follow the ordinary distributative and commutative laws of multiplication, that is:

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} = \mathbf{b} \cdot \mathbf{a} + \mathbf{c} \cdot \mathbf{a}$$

As an application of the dot product we take the proposition from the third book of plane geometry that the square of the median of a triangle equals half the sum of the squares of the two including sides less the square of half the third side.




Let the sides of the triangle be of lengths $a$, $b$ and $c$ and the median $m$ as shown in the figure. Then we have

$$\mathbf{m} = \mathbf{b} + \tfrac{1}{2}(\mathbf{a} - \mathbf{b}) = \tfrac{1}{2}(\mathbf{a} + \mathbf{b})$$

$$\mathbf{m}\cdot\mathbf{m} = \tfrac{1}{2}(\mathbf{a} + \mathbf{b})\cdot\tfrac{1}{2}(\mathbf{a} + \mathbf{b})$$

$$= \tfrac{1}{4}(\mathbf{a}\cdot\mathbf{a} + 2\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{b})$$

or $$m^2 = \tfrac{1}{4}(a^2 + b^2) + \tfrac{1}{2}\mathbf{a}\cdot\mathbf{b}$$

Now $$\mathbf{b} = \mathbf{m} - \tfrac{1}{2}\mathbf{c}$$

$$\mathbf{a} = \mathbf{m} + \tfrac{1}{2}\mathbf{c}$$

$$\therefore \mathbf{a}\cdot\mathbf{b} = (\mathbf{m} + \tfrac{1}{2}\mathbf{c})\cdot(\mathbf{m} - \tfrac{1}{2}\mathbf{c}) = \mathbf{m}\cdot\mathbf{m} - \tfrac{1}{4}\mathbf{c}\cdot\mathbf{c} = m^2 - \tfrac{1}{4}c^2$$

so that substituting this value we have

$$m^2 = \tfrac{1}{4}(a^2 + b^2) + \tfrac{1}{2}m^2 - \tfrac{1}{8}c^2$$

or $$m^2 = \tfrac{1}{2}(a^2 + b^2) - (\tfrac{1}{2}c)^2$$

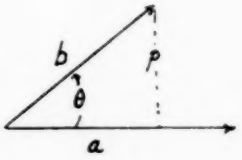
Q. E. D.

Obviously, many problems and theorems involving length of lines can be treated in this way.

The *cross* or *vector product* of two vectors (written $\mathbf{a} \times \mathbf{b}$) is a vector quantity of absolute magnitude equal to the product of the absolute value of $\mathbf{a}$ and of the projection of $\mathbf{b}$ upon a perpendicular to $\mathbf{a}$ and of direction perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$. Thus:

$$\mathbf{a} \times \mathbf{b} = \mathbf{a}_a p\mathbf{k} = \mathbf{a}_a \mathbf{b}_a \sin\theta\mathbf{k}$$

where $\mathbf{k}$ is a vector of unit length perpendicular to the paper downward or in a direction opposite that of the motion of a right-handed screw if a line across its head be turned across the angle $\theta$ from the direction of $\mathbf{a}$ to that of $\mathbf{b}$. We have then, that




$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

that is, if we change the order of the factors we change the sign of the cross product. Again $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ is not equal to $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ so that the commutative and associate laws of algebraic multiplication do not hold in the case of the cross product. But, the order remaining the same, the distributive law of algebraic multiplication does hold, thus:

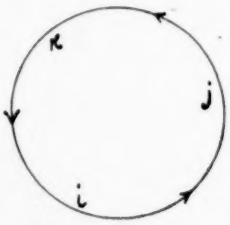
$$\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c} = -(\mathbf{b}+\mathbf{c})\times\mathbf{a} = -\mathbf{b}\times\mathbf{a} - \mathbf{c}\times\mathbf{a}$$

Many problems are simplified by using for reference three mutually perpendicular unit vectors **i**, **j**, and **k**. In this case since $\theta = 90°$, we have

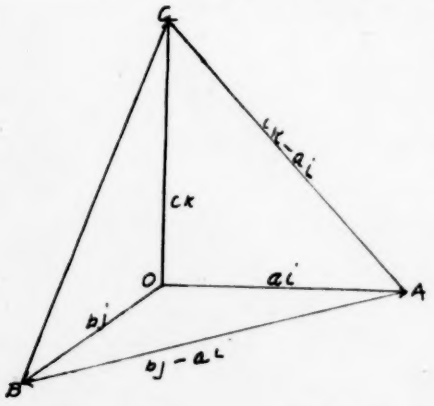
$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1$$
$$\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0$$
$$\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = 0$$
$$\mathbf{i}\times\mathbf{j} = -\mathbf{j}\times\mathbf{i} = \mathbf{k}$$
$$\mathbf{j}\times\mathbf{k} = -\mathbf{k}\times\mathbf{j} = \mathbf{i}$$
$$\mathbf{k}\times\mathbf{i} = -\mathbf{i}\times\mathbf{k} = \mathbf{j}$$

The last three of these results can be read cyclicly from a diagram. In the direction of the arrow heads we get positive products, the other way negative, *i. e.*,

$$\mathbf{i}\times\mathbf{j} = \mathbf{k} \text{ but } \mathbf{k}\times\mathbf{j} = -\mathbf{i}.$$




Since the absolute value of the cross product is equal to the area of the parallelogram whose sides are $a$ and $b$, it is often useful in problems or theorems involving areas. From page 329 of Chauvenet's *Solid Geometry* we have the original: If one of the triedral angles of a tetraedron is trirectangular the square of the area of the face opposite to it is equal to the sum of the squares of the areas of the three other faces.




Let the three mutually perpendicular edges be of length $a$, $b$ and $c$, then using unit vectors we have $\overrightarrow{OA} = a\mathbf{i}$, $\overrightarrow{OB} = b\mathbf{j}$ and $\overrightarrow{OC} = c\mathbf{k}$, and for the edges $AB$ and $AC$ opposite the trirectangular angle

$$\overrightarrow{AB} = b\mathbf{j} - a\mathbf{i};$$
$$\overrightarrow{AC} = c\mathbf{k} - a\mathbf{i}.$$

Then

$$\begin{aligned} 2 \text{ Area } \triangle ABC &= [(b\mathbf{j} - a\mathbf{i}) \times (c\mathbf{k} - a\mathbf{i})]_o \\ &= [bc\mathbf{i} + ac\mathbf{j} + ab\mathbf{k}]_o \\ &= \sqrt{(bc\mathbf{i} + ac\mathbf{j} + ab\mathbf{k}) \cdot (bc\mathbf{i} + ac\mathbf{j} + ab\mathbf{k})} \\ &= \sqrt{b^2c^2 + a^2c^2 + a^2b^2} \end{aligned}$$

Therefore

$$\begin{aligned} (\text{Area } ABC)^2 &= (\tfrac{1}{2}bc)^2 + (\tfrac{1}{2}ac)^2 + (\tfrac{1}{2}ab)^2 \\ &= (\text{Area } OBC)^2 + (\text{Area } OAC)^2 + (\text{Area } OAB)^2 \end{aligned}$$

Q. E. D.

The value of the principles here set forth is more real than is apparent; for the student having a thorough grasp of them can not only solve many problems in elementary mathematics, but is equipped to pass by easy stages through most of the analyses of analytic geometry, differential and integral calculus, analytic mechanics and differential geometry besides much of physics and electrical theory.

## A GREAT MATHEMATICIAN AS A SCHOOL BOY

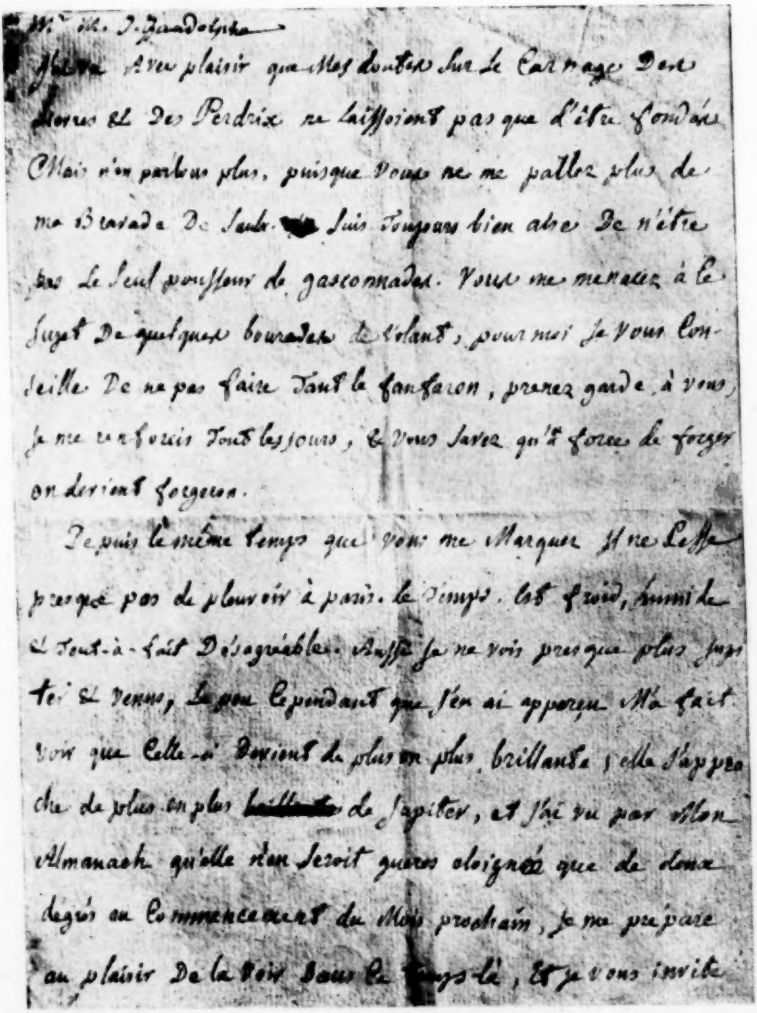
By DAVID EUGENE SMITH and VERA SANFORD

There is always and everywhere present in the human mind the tendency to hero worship. Iconoclastic as we may conceive ourselves to be, theoretically regicidal as we may proclaim our intentions, radical as may the group of which we are members boast itself, we all admire real ability and we tend to bow down before it. This is the reason why we exalt, even unduly, those whose genius we admire, placing them upon pedestals and considering that human frailties are alien to their nature. To us they are heroes ever,—born great and never descending to the average human level.

But we know that this tendency to hero worship is not warranted; that all humanity passes through the stage of childhood, the great as well as those whose mental caliber is looked upon as small; and that the youth of genius and the youth of the dullard have very many points of contact. Nevertheless, we hold that all stories of Thales and Archimedes, of Newton and Leibniz, and of Laplace and Gauss, must begin only with their several periods of eminence.

It is the purpose of this note to ask one of the great mathematicians of the world to speak for himself, and to give to American teachers a passing glimpse of his own boyhood. The mathematician is Adrien Marie Legendre,—great in the theory of numbers (including the principle of least squares), in the field of elliptic functions, and in the applications of the calculus; a prolific writer upon a variety of minor mathematical subjects, and the one who, with better right than any other man, save Euclid, can be called the father of American geometry as taught in our schools today. It was the textbook which he wrote for the purpose of making geometry more intelligible to the youth of France that, through a translation made two or three generations ago, set the standard that is still maintained in the American schools.

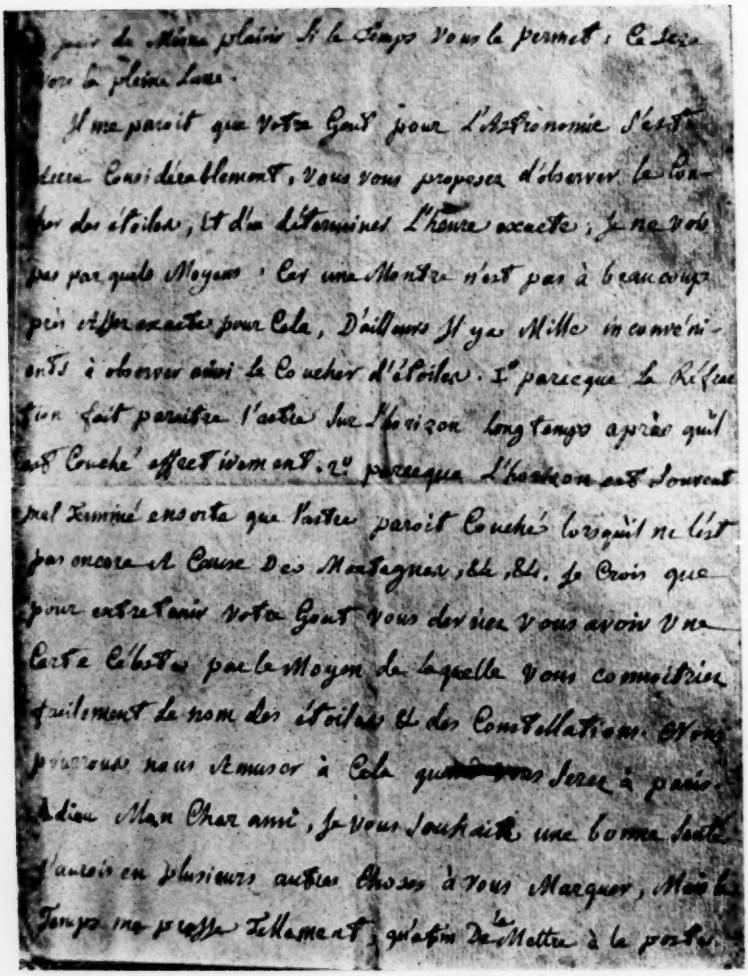
Among the autographs of famous mathematicians which form part of the library of one of the writers of this article, there is a letter in a boyish hand, written when Legendre was only eigh-



M^r de J. Grandchamp

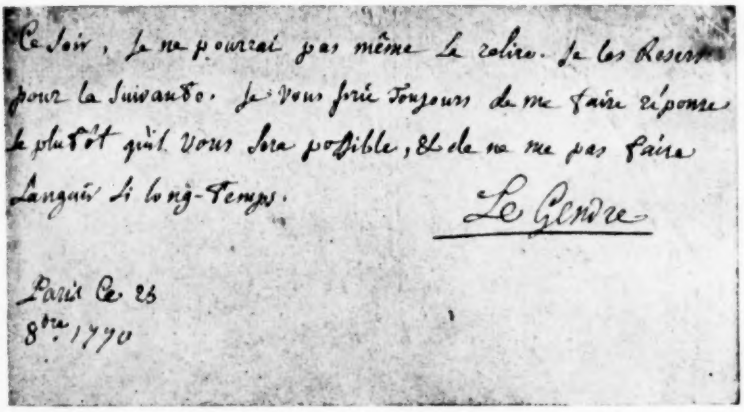
J'ai vu avec plaisir que mes doutes sur le Carnage des Oeufs et des Perdrix ne laissoient pas que d'être fondés. Mais n'en parlons plus, puisque vous ne me parlez plus de ma Bravade. De Sauteur je suis toujours bien aise de n'être pas le seul professeur de gasconnades. Vous me menacez à ce sujet de quelques bourades de volant; pour moi je vous conseille de ne pas faire tant le fanfaron, prenez garde à vous, je me exerce tous les jours, et vous savez qu'à force de forger on devient forgeron.

Depuis le même temps que vous me marquez il ne cesse presque pas de pleuvoir à paris. le temps est froid, humide et tout-à-fait désagréable. aussi je ne vois presque plus jupiter et venus, le peu cependant que j'en ai apperçu m'a fait voir que celle-ci devient de plus en plus brillante, elle s'approche de plus en plus de Jupiter, et j'ai vu par votre almanach qu'elle n'en seroit guères eloignée que de deux degrés au commencement du mois prochain, je me prépare au plaisir de la voir dans ce temps-là, et je vous invite



pour de Mon plaisir Si le Temps vous la permet ; Ce sera
vers la pleine Lune.

Il me paroit que votre Gout pour l'Astronomie s'est
accru Considérablement, Vous vous proposez d'observer le Cou-
cher des étoiles, Et de déterminer L'heure exacte ; Je ne vois
pas par quels Moyens : Car une Montre n'est pas à beaucoup
près assez exacte pour Cela, D'ailleurs Il y a Mille inconvé-
nients à observer ainsi le Coucher d'étoiles. 1° parcequ'la Réfrac-
tion fait paroitre l'astre Sur l'horizon longtemps après qu'il
est Couché effectivement. 2° parcequ'l'horizon est Souvent
mal terminé ensorte que l'astre paroit Couché lorsqu'il ne l'est
pas encore A Cause des Montagnes &c &c. Je Crois que
pour entretenir votre Gout vous devriez vous avoir Une
Carte Céleste par le Moyen de laquelle vous connoitriez
facilement le nom des étoiles et des Constellations. Nous
pourrons nous Amuser à Cela quand vous Serez à Paris.
Adieu Mon Cher ami, Je vous souhaite une bonne Santé
J'aurois en plusieurs autres Choses à vous Marquer, Mais le
Temps me presse Tellement, qu'afin de Mettre à la poste



Ce soir, je ne pourrai pas même la relire. Je la Reserve pour la Suivante. Je vous prie toujours de me faire réponse le plutôt qu'il vous sera possible, & de ne me pas faire Languir si long-Temps.

Le Gendre

Paris Ce 23
8bre 1770

teen years old and was a pupil in the Collége Mazarin at Paris. So far as known it has never been published, and its human character brings before us vividly the youth of the great French scientist who, only five years later, was professor of mathematics in the École Militaire at Paris, and who, a few years thereafter, ranked with Laplace and Lagrange as one of the great mathematicians of France. He weathered the storm of the Reign of Terror, prospered under Napoleon, reached the highest rank among the learned of his day, and yet died in obscurity and poverty.

The letter, translated with such freedom as is necessary to meet the conditions of a foreign tongue, but with the capitalization and general style of the original, is as follows:

M[r] M. T. Zandolphe

I see With pleasure that My objections To The Slaughter Of the Hares and Partridges have not been Allowed to be forgotten. But I shall say nothing more about it since you no longer mention my Boasting when we were at Saulx.* I Am Always happy when I am not The Only braggart. You are now threatening me with Reference To a boxing match. I advise you, for my part, not To make Such a swagger. Look out for yourself, I am growing more proficient Every day, & you know that practice makes perfect.

Since the time that you Wrote me It has hardly Stopped raining at paris. The Weather is cold, wet, and Altogether Disagreeable. I have scarcely seen Jupiter & Venus. From the little that I have observed them, However, I notice that Venus is growing more and more brilliant, she is drawing nearer and near to Jupiter & I read in My Almanac that she will Be barely two degrees distant from him at the Beginning of

* There is a small town by this name in the Haute-Saône, near Paris.

next Month. I am counting on the pleasure Of seeing her At that time And I beg you to share the treat If Time permits. It will Be about the full Moon.

It seems to me that your Liking for Astronomy has Grown Considerably. You propose to observe the Setting of the stars And to determine The precise hour; I do not know by what Means. For a Watch is not sufficiently exact for That. Besides, There are a Thousand hindrances to observing the Setting of the stars. 1° The Refraction makes the star appear On The horizon Long after it has actually set. 2° The horizon is Often badly Defined Because of Mountains &c so that the star appears to have Set when it has not &c &c. I Believe that to cultivate your Hobby you ought to have a Celestial Chart by Means of which you might easily learn The names of the stars & of the Constellations. We can Amuse ourselves with One when you Are in paris.

Adieu. My Dear friend, I wish you the best of Health. I would write several other Things, but I am So pressed for Time that if I am to Put This in the mail This Evening, I cannot even re-read it. I will Save them for my Next letter. I Always beg you to reply As soon as Is possible So that I may not have to Wait Long to hear from you.

LE GENDRE

Paris le 25
8 bre 1770

For a boy of eighteen, such a letter shows great promise,—a promise which, as we know, was abundantly fulfilled. It would be interesting to know how many boys, in the United States today, would be apt to write a similar letter, and how many will leave such a name in the history of the world.

It will be noticed that the name is signed Le Gendre, whereas we commonly see it in the form Legendre. In his youth he always used the former style, but in later life he was in the habit of writing the two parts together, and gradually the capital G became smaller. There was always, however, a trace of the old form, although in print the name appears as it is now commonly known.

# THE FORMULA IN NINTH GRADE ALGEBRA

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*Algebra is a general science.* In elementary mathematics we recognize a body of mathematical material which we are accustomed to classify as arithmetic and another which we classify as algebra. Yet upon examination of the material of these two subjects we find many common problems. Does this mean that there is no sharp line of demarcation between these two subjects? In order to answer this question let us first consider some simple examples.

Let us take for our first example the problem of finding the simple interest on \$500 at 6% for a period of 2 years. The interest is obtained by multiplying \$500 by .06 and the resulting product by 2. That is

$$i = \$500 \times .06 \times 2.$$
$$i = \$60.$$

This problem is characterized by the fact that we are operating on particular numbers with a view to obtaining the interest in this particular case. This is a problem of arithmetic.

But let our attention be directed to the fact that in finding the interest in all cases in which the principal, rate, and time are given we obtain the product of these three elements. Now we are in the field of algebra.

This illustration is given in order to point out the fact that in arithmetic we are interested in particular numbers as we move toward results, while in algebra we direct our attention to the operations required to reach these results and shut out from consideration any particular characteristics of the numbers employed in the operations. In arithmetic we are interested in the combination of a set of particular numbers. In algebra we are interested in the rule for the combination of such sets. Thus arithmetic is a particular science, while algebra is a general science.

*Meaning and use of the formula.* Let us return to the rule for finding simple interest. It may be stated as follows: The

simple interest on a sum of money at a given rate for a given number of years is equal to the product of the principal, rate, and the number of years. Now let us agree that $i$ shall mean the phrase, "The simple interest on a sum of money at a given rate for a given number of years," and $p$, $r$, and $t$ mean principal, rate, and number of years respectively, then the rule could be put in the form

$$i = p\,r\,t.$$

This shorthand, or shortmind,[1] form of expression of this rule is called a formula. A formula may thus be defined as a symbolic expression of a rule or principle. This symbolic mode of expression is another mark of algebra which distinguishes it from arithmetic.

Since algebra deals with generalizations expressed in symbolic language it seems that the pupil could be introduced to formal algebra quite naturally by means of the formula. In his arithmetic work he has been given various rules for the solutions of problems. He is therefore ready and will be delighted to translate these rules into formulas.

Let us consider a very simple example for illustrating the method of proceedure in making his first translations.

The number of feet in a given number of yards is equal to three times the number of yards.

It would probably be well in leading the pupil to the extreme symbolic form to first write it in an abbreviated form as follows:

No. of ft. in a given no. of yd. $= 3 \times$ no. of yd.

So far he sees abbreviations with which he is familiar. At the same time he sees an economy in the time and space required to write it, while there is a saving in mental effort in reading it.

Now let us agree that $f$ shall mean "the number of feet in a given number of yards" and $y$ shall mean "the number of yards." Our abbreviated form then becomes

$$f = 3 \times y.$$

Now if we make the agreement that a number placed adjacent to a letter shall mean the product of the two we can finally write

$$f = 3y.$$

[1] See T. J. McCormack, "Why Do We Study Mathematics?"

Thus a statement of nineteen words is expressed by means of four symbols. It is much more easily read than the original statement and the relation between feet and yards can be seen at a glance.

The introduction of letters to represent mathematical phrases obviates the difficulty the pupil has in the older algebras in which he is confronted with the statement "Let $x$ represent a number." He immediately protests that he cannot see how $x$ can represent a number unless it be some particular number. But in the formula $f = 3\,y$ he readily sees that $y$ stands for the phrase "number of yards" and that this number may be any one he chooses to make it.

With such an introduction to algebra considerable practice should be given in translating from ordinary language to the symbolic form, and conversely. Later I shall give examples of rules and formulas which are suitable for the ninth grade work.

The importance of the formula in mathematics can hardly be over emphasized. Formulas are found in engineering and technical journals, in works of science, in business, in a vast number of trades and professions in which a mathematical statement or investigation is required.

The importance of the formula comes from the fact that it is a great economizer of space, time, and mental effort. By means of it one can disengage his mind of all sensory experiences and can concentrate his attention wholly upon the relation or operation expressed by it. As an illustration consider the expansion of the fifth power, let us say, of the sum of two numbers, viz.,

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

It would require more than one hundred words to express this formula in ordinary language. Moreover it would require much more mental exort to comprehend this relation in this form than in the symbolic form.

The great advances made in mathematics date from the time of the introduction of symbols. One has only to consider such a statement as Taylor's Theorem, viz.,

$$f(x + h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^2}{3!}f''(x) + \ldots\ldots$$

to see the necessity of a symbolism. This relation stated in ordinary language would be bewildering and probably useless for practical purposes.

*Gradation of formulas.* It is needless to say that formulas in a course in algebra should be arranged in the order of their complexity. However people might differ as to the position in which any particular formula should be placed. In the course which I have developed I have made the function the organizing principle. The functions which I have used are integral algebraic functions of the first and second degrees. I have arranged them in the following order.

1. $y = ax$
2. $y = ax + b$
3. $z = axy$
4. $y = ax^2$
5. $y = a(x^2 - b^2)$
6. $y = a(x + b)^2$
7. $y = ax^2 + bx + c$

Now a formula may be looked upon as a relation between variables. If one of the variables in a formula is expressed in terms of the others it is said to be a function of these variables. Thus in all of the relations above, excepting (3), $y$ is a function of $x$; while in (3) $z$ is a function of $x$ and $y$. If, therefore, a formula can be put in either of the forms listed it should be placed in the course under the function to which it belongs.

*Formulas should be interesting.* No formula should be given a pupil unless it concerns something of interest to him. Whenever convenient it should be derived by him. Thus the formula,

$$F = \tfrac{9}{5} C + 32,$$

has much more interest and meaning for the pupil who has actually worked out the relation between the two thermometer scales, who has expressed this relation in good English, and who, finally, has translated his statement into this shorthand form.

There is a vast amount of material which may be used in formula construction. The rules of arithmetic may be drawn upon. Experimental geometry offers a rich field. Here we

can find formulas on perimeters, similar figures, and areas. Then there are many mechanical and physical laws within the easy comprehension of the pupil. Among these are uniform motion, forces on an inclined plane, motion on an inclined plane, motion under constant acceleration, vibration of the simple pendulum, and distance traversed by a body moving under constant acceleration. Many business rules may also be expressed by formulas falling within the range of the ninth grade work.

*Formulation and translation.* No attempt should be made by the pupil to construct a formula until the situation he wishes to express by it has been carefully analyzed and formulated in words. As a simple example consider the construction of the formula for the area of a rectangle. After the pupil has learned that the area of a rectangle means the number of square units that will cover it, let him be required to find the area of a rectangle, 8 by 11 inches, let us say, which has been divided into square inches without counting the individual units. He soon discovers that this may be done by counting the number of squares in a row and multiplying the result by the number of rows. He is then ready, so far as the case of integral dimensions is concerned, and should be required to state that the area of a rectangle is equal to the product of the base and altitude. Now he may translate this into the formula,

$$A = ab$$

Throughout the course the pupil should have much practice in the formulation of mathematical statements in words and the translation of these statements into formulas. Conversely, he should be required to translate formulas into ordinary language and to see that an elegant and precise rendering has been made.

*Some Examples of Material to be Used.* I shall now add some typical examples which I am accustomed to give my classes. They are adapted to ninth grade work and are arranged in the order in which they are presented.

1. Write a formula for finding the perimeter of a square when the number of feet in the length of a side are given $(p, s)$.

Before writing this formula the pupil should be required to state the rule in words. The notation $(p, s)$ signifies that he is

to let $p$ mean "the number of feet in the perimeter" and $s$, "the number of feet in the length of the side." The advantage gained by having pupils use the same notation for purposes of comparison is obvious.

2. Motion is frequently transmitted from one piece of machinery to another by means of a belt. The safe working strength, expressed in pounds, of a belt per inch of width is equal to 300 times the belt expressed in inches. Express this statement by means of a formula $(s, t)$.

3. A rule used by some dairymen in feeding grain to dairy cows is as follows: To determine the number of pounds of grain to feed a dairy cow per day divide the number of pounds of milk produced by her by 3.5. Write the formula $(g, m)$.

4. A train travels 40 miles per hour. Let $d$ mean distance in miles, and $t$, time in hours, then

$$d = 40t.$$

Express this formula in words.

5. A formula which will give the time between any two consecutive hours when the hands of a clock are together is

$$m = \tfrac{60}{11} h$$

where $m$ is the number of minutes past the hour and $h$ is the number of hours. Translate.

6. With an inclined plane, car, and spring balances from the physics' laboratory it is easy to have the pupils see that

$$\frac{p}{w} = \frac{h}{l}$$

where $p$ and $w$ mean power and weight, and $h$ and $l$ mean height and length of the plane, and provided the power acts in a direction parallel to the plane.

From this relation pupils can find

$$p = w \cdot \tfrac{h}{l}$$

or $$p = w \sin i$$

where $i$ means the angle of inclination.

Of course this formula presupposes familiarity on the part of the pupil with the trigonometric ratios.

8. It is not difficult to lead a class to see that velocity on an inclined plane is given by the formula

$$v = 32tsini.$$

The formulas dealt with above are of the form

$$y = ax.$$

A vast amount of material giving rise to relations of this form is available. The formulas which follow immediately fall under the more general form

$$y = ax + b.$$

9. A recipe for making coffee is the following. Allow one tablespoonful for each cup and one for the pot. Translate this rule into a formula $(t, c)$.

10. I bought a set of books for which I was to pay \$18. I agreed to pay for them at the rate of \$1 per month. Write a formula expressing the amount still owed at the end of a given number of months $(a, t)$.

11. A certain society charges \$10 for the initiation and \$5 per year for dues. Write a formula for computing the amount paid in by a member after any number of years $(a, t)$.

12. The Chicago Telephone Company's rate for a four-party line is as follows: A guarantee of 5 cents per day, *i. e.*, \$1.50 per month, plus 4 cents for each outgoing message above 30. Write a formula for finding the cost for any number of messages $(c, m)$.

13. Given a table of rates of postage for the various zones the pupil can easily construct formulas giving the postage for each zone on any number of pounds. These formulas are of the form

$$p = f + c(w - 1).$$

14. As a train increases its speed the resistance to its motion, due to the air and other things, increases. The following formula has been suggested for computing this resistance, viz.,

$$r = 2 + \tfrac{s}{4}$$

where $r$ means the number of pounds of resistance per ton of load and s, the speed in miles per hour.

Translate this formula into a rule for computing resistance.

15. The formula for computing the postage on a parcel directed to a local point is

$$p_0 = .05 + .01\,{}^{w}\!\!\div\!{}_{2}{}^{1},$$

where $p_0$ is expressed in dollars and $w$ in pounds. Translate.

16. The swimming tank in Hyde Park High School has a capacity of 54,000 gallons. It can be filled at the rate 1,500 gallons per hour. At a certain instant while it is being filled it contains 22,500 gallons. Write a formula by which the number of gallons in the tank may be computed for any time before or after this instant $(g, t)$.

The formula is

$$g = 22{,}500 + 1{,}500t.$$

$t$ may take both positive and negative values. $g$ is always positive.

17. A dairyman, who retailed his milk, derived a formula by which he could compute the approximate annual profit on a cow if he knew the average number of pounds of milk she gave per day. The formula was

$$p = 15m - 135,$$

where $p$ is the profit in dollars and $m$ is the average number of pounds of milk per day

Notice that $p$ may take both positive and negative values while $m$ is always positive.

18. When weights of 2 ounces, 6 ounces, 8 ounces, and 12 ounces are suspended from a rubber cord, its lengths are found to be 7 inches, 9 inches, 10 inches, and 12 inches, respectively. Write a formula for computing the length for any other weight $(l, w)$.

19. A railway station, $B$, is 190 miles east of a station, $A$. A train running on a road passing through $A$ and $B$ is leaving $B$ and going east. Distance east of $A$ and velocity east are considered positive. If the train runs at an average rate of 40 miles per hour, what will be the formula giving the distance, $d$, from $A$ at the time, $t$, time being counted from the instant that the train leaves $B$.

The formula is

$$d = 190 + 40t.$$

This is an excellent formula for giving practice in the interpretation of directed numbers.

20. Write a formula for computing the $n^{th}$ term of the arithmetic progression,

$$\cdots\cdots\cdots 2,\ 7,\ 12,\ 17,\ \cdots\cdots\cdots$$

starting from 12.

21. Two trains are now $d_o$ miles apart and are traveling on a straight road at the rates of $r_1$ and $r_2$ miles per hour. Write a formula giving their distance, $d$, apart at any time, $t$.

The formula may be written as

$$d = d_0 + (r_2 - r_1)t$$

It has a broad application since it can be used in expressing the relation between objects which are changing uniformly either in the same or opposite senses. It offers an excellent opportunity for practice in the interpretation of directed numbers.

22. A boy rolls a ball up a smooth incline with velocity of $v_o$ feet per second. Its velocity decreases uniformly at the rate of $a$ feet per second. Write a formula giving the velocity of the ball at any time $(v,\ t)$.

23. A sea-coast town, $Q$, bears N25°E from another coast town, $P$. A ship is sailing from $P$ to $Q$ at the rate of 18 miles per hour and is now 4 miles directly east of a coast town, $T$. Let distance north and east and rates in a northerly direction be positive. Write a formula giving the departure of the ship from $T$ at other times $(d,\ t)$.

The formula is

$$d = 4 + 18t\sin 25°.$$

Formulas of this type give practice both in the use of trigonometric ratios and in interpreting directed numbers.

24. A point moves so that its distance from the $x$-axis is 3 increased by twice its distance from the $y$-axis. A second point moves so that its distance from the $x$-axis is 6 diminished by its distance from the $y$-axis. Find the co-ordinates of the point of intersection of their paths.

25. Two rubber cords hang side by side suspended from a horizontal support. One is 18 inches long and is stretched $1\frac{3}{8}$ inches by each additional ounce hung at its end. The second

is 21 inches long and is stretched 7/16 inch by each additional ounce. What weight will stretch them to the same length and what will the length be?

26. The pupil will have little difficulty in deriving the formulas for the areas of parallelograms, triangles, and trapezoids, viz.,

$$A = bh, \quad A = \tfrac{1}{2}bh, \quad A = \tfrac{1}{2}h(a + b).$$

27. If two adjacent sides of a parallelogram are $a$ and $b$ and the included angle is $\alpha$, then

$$A = ab\sin\alpha.$$

Similarly for the triangle

$$A = \tfrac{1}{2}ab\sin\alpha.$$

28. From the definition of square formulate a rule for finding its area and express it as a formula $(A, s)$.

29. The load that may be safely carried by an iron chain is given by the formula

$$l = 7.11d^2,$$

where $l$ is the load in tons and $d$ is the diameter of a link expressed in inches. Translate.

30. The time of vibration of a simple pendulum is given by the formula

$$t = .554\sqrt{l},$$

where $l$ is the length of the pendulum in feet and $t$ is the time in seconds. Translate.

31. If the formula giving the velocity of a falling body is

$$v = 32t,$$

show that the distance, $s$, through which it falls in $t$ seconds is

$$s = 16t^2.$$

The derivation of this formula requires the use of the formula for the sum of an arithmetic series.

32. From the definition of annulus show that its area is given by the formula

$$A = \pi(r_1^2 - r_2^2)$$

33. In a "potato" race the potatoes were placed in a straight line at equal intervals of 10 feet. A basket, in which a contestant was to deposit each potato in turn, was placed at a distance of 15 feet from the first potato and in the line of potatoes. Show that the total distance traveled by a contestant is

$$d = 10n^2 + 20n,$$

where $n$ is the number of potatoes.

34. If a body with an initial velocity $v_o$ moves under constant acceleration, $a$, so that its velocity is

$$v = v_o + at,$$

show that the distance passed over during time, $t$, is

$$s = v_o t + \tfrac{1}{2}at^2.$$

Since the letters in this formula represent directed numbers, it offers an opportunity for more practice in their interpretation.

I wish to turn aside here to add a remark in regard to the use of simple material from mechanics. There seems to be a disinclination on the part of some mathematics teachers to use such material on the ground that it takes too much time for the demonstration of mechanical principles, and anyhow they belong to the department of physics.

Of course on the same ground one could object to the use of geometrical material since its takes time to get geometrical principles before the pupils, and anyhow they belong to the course in geometry. On the same ground one might object to introducing any material for the purpose of making applications.

The answer to these objections is obvious. The majority of ninth grade pupils will never take a course in physics. The simple mechanical laws do not require much time for their illustration. Moreover, mathematics is just as much concerned with mechanical laws as it is with geometrical laws or any kind of laws. Our high school mathematics in the past has been formal and, so far as the average pupil is concerned, sterile. Our pupils have asked us to give them a *raison d'etre* of algebra and the only justification we could give was that it possibly trained their minds and, if they continued work in mathematics, they would need it in later courses.

These two answers are really identical. For, if a pupil is given nothing but formal mathematics, his mind will be trained for that sort of mathematics and that only. If we wish algebra to function in the lives of our pupils we must see to it that it has many points of contact with their experiences. There is no transfer of mental training from algebra unless the teacher sees to it that the pupil identifies its laws with the laws of nature and society about him.[1]

*Evaluation of Formulas.* If a formula is written in the explicit form it may be considered as a general solution. To obtain a particular solution we need only substitute the given numerical values of the letters and then simplify. Since the necessity for the evaluation of formulas occurs very frequently not only in mathematics but also in all sorts of scientific and technical work, it is highly desirable that the pupil should be given a great deal of practice in substitution from the beginning of the course.

If a number of particular solutions for a letter in a formula are required it is probably desirable to solve the formula beforehand for this letter and then substitute the given values of the other letters. However, if one meets a problem in which it is required to find only one numerical value of a letter which is not expressed explicitly in terms of the others, it is better to substitute the numerical values of the other leters in the formula and then solve the resulting equation for this letter. Thus a large amount of legitimate equation material may be obtained in this way.

Moreover, in general, it is better that a pupil become familiar with but one form of a formula. For example, it is better that the formula for uniform motion should be learned in the form $d = rt$ rather than in all three forms. Or, that the formula,

$$s = v_0 t + \tfrac{1}{2}at^2,$$

should be known in this form rather than in a number of forms in which it might be expressed.

In solving a problem involving a formula the pupil should be required to write the formula first. On the part of the pupil this guarantees that he think out the plan of his solution at

[1] C. H. Judd, "Psychology of High School Subjects." Especially the chapter on "Generalized Experience."

the start. On the part of the teacher it means a saving of time and energy in discovering the plan of the pupil. After the formula has been written the known value of the letters should be set down. These values should then be substituted in the formula and the resulting equation solved.

Let us consider some examples of equations which arise from substitution in formulas and the form in which the work should be written.

Example 1. What time will be required for a ball to attain a velocity of 22.3 feet per second in rolling down a smooth incline which has an inclination of 6°?

Solution.

$v = 32.2t\sin i$

$v = 22.3$

$\sin 6° = .105$

$\therefore 22.3 = 32.2 \times .105t$

$22.3 = 3.38t$

$t = 6.63$

Computation:

| | | |
|---|---|---|
| 32.2 | 22.30 | 3.38 |
| .105 | 20 16 | 6.63 |
| 3.22 | 2 14 | |
| 16 | 2 03 | |
| 3.38 | 11 | |
| | 10 | |
| | 1 | |

Example 2. Change — 58° F. into degrees C.

Solution:

$$F = \tfrac{9}{5}C + 32$$
$$F = -58$$
$$\therefore -58 = \tfrac{9}{5}C + 32$$
$$-90 = \tfrac{9}{5}C$$
$$-450 = 9C$$
$$C = -50$$

Example 3. An automobile is traveling at the rate of $v_0$ miles per hour when it begins to change its velocity $a$ miles per hour every minute. If the velocity is now 27 miles per hour, the original velocity 15 miles per hour, the change in velocity + 3 miles per hour every minute, how many minutes has it been changing its velocity?

Solution.

$$v = v_o + at$$
$$v = 27$$
$$v_o = 15$$
$$a = 3$$
$$\therefore 27 = 15 + 3t$$
$$t = 4$$

Example 4. A rifle was fired horizontally from the top of a cliff 100 feet above the level of a lake. The bullet had a horizontal velocity of 800 feet per second. How far from the foot of the cliff did it strike the water?

Solution.

$$s = 16t^2$$
$$s = 100$$
$$\therefore 100 = 16t^2$$
$$t = 5/2$$

$$d = rt$$
$$r = 800$$
$$t = 5/2$$
$$\text{d} = \tfrac{5}{2} \times 800$$
$$d = 2000$$

*Solving a Formula.* Since it is sometimes desirable to express one of the letters of a formula explicitly in terms of the others and since, also, a change in the form of a formula may give additional information, the pupil should be given practice in solving formulas for any letter. If the steps in the solution of a numerical case are noted, it becomes a simple matter to write down the corresponding general solution. Thus if the pupil is required to solve the formula,

$$F = \tfrac{9}{5}C + 32$$

for $C$, let him first find the value of $C$ let us say for $F = 68$ without making any combinations. Thus:

$$68 = \tfrac{9}{5}C + 32$$
$$68 - 32 = \tfrac{9}{5}C$$
$$5(68 - 32) = 9C(68 - 32)$$
$$\text{C} = \tfrac{5}{9}(68 - 32)$$

Now it becomes evident to the pupil that 68 may be replaced by $F$ thus obtaining

$$C = \tfrac{5}{9}(F - 32).$$

After some practice in this proceedure he should be required to solve for letters as if they were numbers.

# COMBINED MATHEMATICS

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The writer has found a good many teachers of high school mathematics who are not clear as to the nature and significance of combined mathematics, or as some term it, fused mathematics. He has further found that a considerable number of these teachers are honestly, though not always very actively, seeking light on the subject. In localities where the adoption of such courses in high schools is imminent, the question is one of considerable interest. In talking with teachers there has been found a wide divergence of opinion as to what constitutes a combined course in mathematics. Even the text books do not appear to convey a uniform opinion on this point. In what follows, the writer, in giving his views, may inadvertently appear to take sides.

It may be well to consider first, very briefly, the origin of combined mathematics from the pedagogical standpoint. How has combined mathematics come into existence? A number of the best and most conscientious teachers recognized the unsuitableness of the traditional courses under existing conditions in school. They at last recognized that Euclid and his successors were not suited to juvenile instruction. Euclid was originally intended for mature adults. A similar conclusion was reached regarding the traditional algebra. These teachers set about constructive remedial measures. They were patient and devoted, and finally became insistent. Combind mathematics is one of the results of their efforts. In so far as their testimony goes, considering their reputation as teachers, it must command serious attention. They had no idea of elimination. They wished to make the courses in mathematics more atractive and more efficient as an educational instrument. No claim of perfection or finality comes from them. We, as teachers, are to try out, and to further improve what they have given us.

One of the greatest difficulties to an understanding of combined mathematics is lack of information. There is, also, the ever present difficulty in the way of every new move, fear. This is psychological and will vanish when the first has been re-

moved. To some, the term combined mathematics, seems to suggest the impossible task of teaching simultaneously, arithmetic, algebra and geometry in some form not very dissimilar to the present traditional courses. To others, the term suggests an undefined "hotch potch." To still another group, the term suggests a sort of materialistic substitute which is not mathematics at all. So it goes on. Now, as the writer conceives combined mathematics, these are all wrong or at least, very misleading.

Taking up the first difficulty mentioned above, it is to be observed that the traditional mathematical curriculum of a standard high school is usually: one year of algebra, one year of geometry, half a year or a year of advanced algebra, followed by solid geometry or plane trigonometry, or both. Plainly, this course is a succession of partial courses or branches taken through the high school period. That is, we have an alternation of topics in rather large units of years and half years. One thing that combined mathematic purposes is to alternate the topics in smaller units, very much smaller units. Thus the difficulty of simultaneity as above indicated is removed.

Then comes the question of the unifying or co-ordinating principle which is to hold the course together. Let us first ask what has been the unifying principle of the traditional courses. In geometry, it has been logical sequence for the most part, and a fair degree of completeness as a final goal. In algebra, it seems to have been the historical order of development of the operations of arithmetic with the generalized numbers, as is the fashion in ordinary arithmetic. This may be termed a cumulative instrumental value of processes. Individually, each process is developed more or less logically. Now what appears to be the unifying principle of combined mathematics? In answer we may say there are several. The author of one text on combined mathematics for college freshmen has used the notion of function as a connecting principle. In fact, each chapter is roughly a more or less complete mathematical treatment of a single type or group of functions. Various operations are applied, in order, to each type. The types increase in complexity as the course proceeds. Another author has apparently taken the cumulative instrumental value as a unifying principle. There is evidence that he has made use of the function idea in

certain parts. Other writers have made use of these and other principles in constructing courses in combined mathematics. Only the most outstanding principles have been pointed out. For a complete analysis would show various subordinate principles woven into the larger plan. To mention all these would take us beyond our limits in this paper.

It seems to the writer, that the function idea, in any of its current forms, would not be well adapted as a unifying principle in courses for first year high school pupils. The principle of cumulative instrumental value seems practicable and more in line with practice in some of the older courses.

It may now be asked whether the logical consistency of the development need be sacrificed when some principle, other than logical sequence, is made fundamental in binding the course together. The answer is certainly in the negative. For there is ample place for the principle of logical sequence as well as scientific proceedure in the development of particular topics. This will be illustrated later. Then what is to be gained by changing from the time honored course to the combined course? Different advocates will offer different claims. To the writer, some advantages stand out clearly.

(1) It is possible to so arrange the course in a sequence that the most useful (both mathematically and practically) can be presented in cumulative order, while the acknowledged difficult parts can be postponed to a later stage, when they will be more easily approached.

(2) By bringing algebraic methods and geometric methods into close relation from the start, a longer period of training in correlating these methods is gained.

(3) For students who will not go far in mathematics, either on account of taste or ability, the combined course seems to offer more that is useful (or usable) and within the grasp of the weaker (mathematically) than is possible under the old plan.

(4) Those who may desire to take fore advanced courses will approach them with a better perspective of their content and purpose. As a consequence, they will make more rapid progress in the more specialized branches, such as formal algebra and demonstrative geometry.

(5) Saving of time has been offered as a distint advantage. To the writer it seems this may be open to question. But teachers who have tried it claim that both plane and solid geometry can be done satisfactorily in a single year, following a good course in combined mathematics. They also claim that formal algebra can be covered through "third semester algebra" in a single year, after combined mathematics. This appears to be entirely feasible. It appears that in three years, pupils beginning with a year of combined mathematics can get as far in formal mathematics as under the old plan in the same time. They will probably have a much better use of their mathematics at that stage under the new plan. It remains to be seen whether there can be an actual material shortening of the time to cover three years' work. The fact remains that no time is lost by the new plan.

It may be suggestive to outline an example of proceedure for the sake of definiteness. For example, one book starts out with the use of the simple equation as an instrument in solving concrete problems within the understanding of the pupils. Incidentally, but with definite purpose, through the first chapter some of the terms and fundamental processes relating to equations are used and explained, such as:

1. Equal numbers may be added to both members of an equation.

2. If both members of an equation be multiplied by the same or equal numbers, the results will be equal.

In the next chapter, some definite fundamental geometric facts and principles are introduced. Then the equation is applied to a new set of problems involving the geometric knowledge just gained.

In a following chapter, some further algebraic and geometric principles are developed and applied to problems of a more advanced type. In general, the entire year's work is developed systematically and with as much logical soundness as the pupils are capable of appreciating. There is thus accumulated a stock of intuitional and experimental geometry correlated with a working knowledge of the fundamentals of algebra in the use of simple forms. Such a foundation will, with little doubt, form

a good basis for the study of a standard course in formal algebra or geometry.

It is to be remembered that the claims above mentioned are dependent upon good teaching. It follows that a responsibility rests with teachers for the success of the plan. It must be remembered that the old plan has been discredited, rightly or wrongly, and cannot be restored if the new plan should fail. The question comes, "What then?" The answer is, *no required mathematics* in the high school curriculum. Such a result would be little short of a catastrophe in education, and while it probably could not be permanent, it would take a generation to impress the fact upon the public, and that much time would be lost. The logical thing to do is to make the revised plan a success as far as possible.

Many teachers object to combined mathematics on the ground that it is not sufficiently thorough. What is thoroughness? Is it not possible that some confuse thoroughness with completeness, in the sense of covering all that is available in a subject? Thoroughness in elementary instruction cannot be measured by completeness. It must be measured by the degree of fixedness of that which is attempted. A course in algebra covering only the simple equation is, so far as concerns the solution of concrete problems, or may be, thorough. This much is done thoroughly when pupils can use such knowledge well and readily. Such a course would not be a complete course in algebra, however. We will say, then, that a course is thorough if it trains pupils well in the use of a set of methods and principles, even if limited to simple matters. The thoroughness here advocated lies more in training in the use of scientific proceedure. It is believed that herein lies the educational value of mathematics, if there be any beyond the few facts needed in daily life.

In closing, a brief outline of a method of scientific proceedure will be given. A similar one has been given during the last several years by the writer to his classes in the "Theory and Methods of Teaching Secondary Mathematics."

Reasoning may be defined as a rearranging and a recombining of ideas and facts with a definite purpose. In all reasoning it is necessary to have a definite starting point, to understand the meaning of all terms used in the given data and in the processes

of rearranging the ideas. It is further necessary to have definite rules, or criteria of proceedure. To be specific:

1. The given data, conditions or hypotheses must be clearly known.

2. The objective or conclusion (thing to be done) must be known.

3. Every term used must be defined (understood).

4. There must be an analysis of the given conditions and of the objective, with the view to finding a set of facts, ideas or steps leading from the data to the objective.

5. Assemble all facts and ideas that are known and that seem to be related to the problem.

6. Select a succession of steps or relations that will lead in an orderly, convincing way from the data to the objective.

7. When a selection of steps has been found and tried, and found unsatisfactory, another set must be sought and tried out.

8. Having found a set of steps that is satisfactory, state the result and interpret it, indicating any special cases or restrictions that are to be observed.

The question arises here as to whether we have given as much effort to teaching scientific methods of proceedure as to teaching mathematical facts. Should not every new problem be made a means of cultivating the habit of scientific methods of proceedure?

Without burdening this paper with further details, the writer can hardly do better than to refer the reader to *Fundamental Aspects of Mathematical Training,* by Prof. Sanders, Bulletin of Louisiana State University, Vol. XIII-N. S., February, 1921, where an excellent collection of carefully worked out illustrations may be found.

# TEACHING PUPILS HOW TO STUDY MATHEMATICS

By ALFRED DAVIS
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## II.

So far the suggestions which concern study and teaching have been of a general nature, applicable, for the most part, to any study. We shall now discuss in greater detail the study of geometry. We select this subject because it offers unusual opportunities to both teacher and student for both teaching and study. The suggestions made for geometry will make clear the proper methods of procedure for other mathematical studies. Moreover, the student who handles geometry successfully is probably equipped to study any subject to which he wishes to apply himself.

Some teachers start a class in geometry by spending the first few days in the reviewing of algebra. This is a serious mistake. It matters not how expert the pupil may be in algebra, nor how well he may like the subject, such a procedure at the beginning is a distinct loss. The pupil comes with an appetite for a new subject, and he is curious to know what that subject will be like. The wise teacher will use this advantage to begin the study of geometry at once. The teacher who fails to do this raises unnecessary barriers to the pupil's interest and success—a disadvantage that, in some cases, may never be overcome. If the review of algebra should be necessary, let it come at the time the subject is needed in the geometry.

The work in geometry should not begin with formal demonstrations. To plunge into this is to bewilder and to discourage the pupil. He cannot recall anything like these proofs in his experience; and so it is impossible for him to make the proper associations whereby the new material can be related to what is already known. Accordingly, for a long time (perhaps for all time) the study means nothing to the student. It does not help matters for the teacher to tell the pupil that he subject will clear up later, the chances are that the prospect for this has been already lost in the dislike of the student for the subject. It is usually such procedure at the beginning of the course that leads

pupils to say, "Geometry is hard," or "I hate geometry." The difficulties of a pupil of normal ability are more than likely due to mistakes in teaching.

In this day of enlightenment and of progress it may be safe to assume that no teacher will begin geometry by assigning long lists of definitions and axioms to be memorized without any apparent reason therefore. These should be learned only when needed for further progress in the study of the subject. At such times it is convenient to refer to them as tools with which to work, just as the carpenter needs to know the use of the hammer, saw, etc., before he can build anything worth while, so we must be thoroughly familiar with certain tools before we can build our interesting proofs. Since these definitions, which are the fundamental hypotheses for our work, are justified by experience, we must see to it that the experience of the pupil is extended, if necessary, to enable him to thoroughly understand their meaning and to feel justified in accepting them. It often happens that what is perfectly obvious to the teacher, is "Greek" to the pupil. A constant effort must be made to ascertain what the pupil knows, for it is only as we build on this as a foundation that our efforts can have any value whatever.

The introduction of a class to the study of geometry is one of the most important parts of the course. Success or failure frequently depends on this. The subject should be made interesting from the beginning. The teacher should question the class as to what their notions of geometry and its values are. With this as his starting point he should lead them to a general survey of the ground to be covered, and the advantages to be sought in the study. The subject should be given an historical setting. The part it has played in great engineering efforts, its importance in further progress, and the invincibility of its conclusions will appeal to the pupil. The table of contents in the text may be used as a guide for the student when giving a survey of the progress of the study.

Having stirred the pupil's interest and expectation the pupil should be provided with some simple instruments such as ruler, compass, protractor, etc. Such exercises as the following should be worked out neatly by the pupil under suitable direction:

Bisect a line.

Bisect an angle.

Draw a perpendicular to a line at a point on the line, at a point not on the line, at the end of the line.

Circumscribe a circle about a triangle.

Inscribe a circle in a triangle.

Inscribe a hexagon, a square, etc., in a given circle.

Make some interesting designs, some of which may be devised by the pupil.

It will be desirable to have the pupil put these constructions neatly in a note-book. This note-book may later contain the most difficult and most interesting exercises of the course, and such matter as the pupil needs for reference, as well as exercises and supplementary work assigned from time to time by the teacher.

This work will acquaint the pupil with geometric forms, constructions, definitions, etc.; develop accuracy that will be important in later work; and above all it will maintain interest, enthusiasm, and appreciation. Formal proofs of these exercises should not be attempted at this time. However, the pupil should check his work by measurement, and when he can he may, in an informal way, say why he thinks the construction ought to be accurate. This will lead, in the case of some constructions, to a realization for the need of a proof, and the pupil will have an appetite for the more formal proof to come later. The minimum time for this sort of work is about a week; the work should not be allowed to drag; if interest cannot be maintained in one thing try something else, keeping in mind the inexperience and limitations of pupil.

It is the utmost folly to begin our proofs by trying to prove something that seems perfectly obvious to the pupil. The pupil must realize the need for a proof, or the work will be trivial and unimportant for him. Those propositions which the experience of the pupil justifies him in accepting should be postulated—assumed true. It may be that he will later see need for the proofs and then they may be taken up again. The following scheme has been tried with success. The teacher goes over the statements of the theorems carefully with the class. If every member of the class is willing to accept the theorem without proof it is indicated on the blackboard in the column for the-

orems not to be proved; if all think, or if any member of the class thinks, a proof is needed, the theorem is indicated in the column for those needing proof. As the game progresses the class may transfer some that they at first thought did not need proof to the other column. All those which finally remain in the assumed class are assumed, at least for the time being. Of course, the success of such an experiment depends on the skill of the teacher. The pupil is likely to assume the truth of such theorems as: All right angles are equal. All straight angles are equal, etc. If the pupil is inclined to assume too many theorems, a few exercises may be used to convince him that looking at a figure is not always sufficient evidence for an assumption. Considerable interest may be aroused by having a number of pupils who have seen the same event report on the details of the occurrence, and noting the contradictions that come up. Two parallel lines may be drawn on the board, and these crossed by numerous other lines drawn through a point midway between them. For most pupils the lines will now seem bent and no longer parallel. Other devices to be found in any modern geometry text may be used. The pupils are usually surprised and interested with these experiments and will be led to question some of the statements which seemed obvious before. Again, if the teacher wishes to start with a theorem which the class does not see the need of proving, it may be studied as a model of what a proof ought to be, and as an easy example of this. If two straight lines intersect, the vertically opposite angles are equal is such a theorem. Indeed all the theorems worked out in the book should be treated as models, and the exercises should be treated as theorems upon which the pupil is to apply the methods learned from the models and so measure his own progress. Otherwise, all the proved theorems of the text might be learned without mastering geometry.

The necessary parts of a proof may be approached and illustrated by reference to debates, with which the pupil is probably somewhat familiar. In this way he will learn the meaning of a proof in geometry, and he will early appreciate its clear, concise, and accurate statements, and the certainty of its conclusions. He will rapidly acquire confidence in his own ability to think. The study of geometry should be directly related to

the study of debate, not only by the teacher of geometry, but also by the teacher of debating.

It is usually a good plan to require only informal and oral proofs during the first weeks. To insist on completely written proofs from the beginning is to make the work unnecessarily difficult, and to discourage the pupil. Time is needed to become acquainted with this new method of reasoning, and it should be our first concern to have the pupil feel comfortably at home with the process. The pupil's attention may, at the proper stage, be directed to the importance of being able to write his arguments in a convincing manner. A few attempts at this, followed by a thorough and constructive discussion will lead to the mastery of the formal proof. The teacher may aid the pupil greatly by working out the proof of an "original" with the class and by writing it on the board. This exercise may then be erased from the board and assigned for home work. If the results are satisfactory other "originals" may be assigned to be worked out by the pupil without assistance. It would, however, be drudgery to require that all originals be written out in this laborious manner. Such a requirement would rob the best part of geometry of much of its interest. Great care should be exercised by the teacher in the selection of problems. No text has yet been so well devised that all the problems it contains should be worked by any one class, nor does any text contain all the problems that a teacher may use to advantage. Reasonable care should be given to accuracy in the drawing of the figure used, and to the form of the written proof. It is not always desirable, however, to insist on a set form of written work for every member of the class. The class must be taught to analyze a problem for the purpose of discovering proofs for themselves.

The writer uses the following method with much success: After the class has become somewhat accustomed to the solving of exercises, anticipate a new theorem by giving the statement of it to the class, but keep the fact that it is another proved theorem a secret. Discuss the new problem and by skilful questioning, where necessary, develop the proof. When the work has been completed the class will be surprised and elated to learn that they have proved the next theorem, and it will usually seem an easy matter, besides teaching concretely how to study a ge-

ometry lesson. The pupils will read the proof for this theorem as given in the text with great interest to discover if possible any variation from their own proof. If any differences can be found the pupils can be impressed with the fact that they are studying a department of knowledge and not merely a text. They will read the text more carefully thereafter, and try to find proofs other than those given. Of course pupils should be encouraged to find new proofs. They should be told if this is necessary, that such work counts more to their credit than any amount of reproducing of what is given in the text. The method just suggested is applicable early in the work and for especially difficult theorems as well as for all exercises that give trouble. It will not do to anticipate every advance lesson, or the interest of the class, and its motive for effort, will be destroyed. The pleasure from the study of geometry consists in the mastery of difficult problems without help from anyone, and in learning how to gain this mastery.

The interest of the pupils may be stimulated by the practical application of geometry where this is convenient. The pupil may be assigned the making of simple instruments, and these may be used in working out problems in the classroom or out-of-doors. For example, the pupil may make a quadrant and use this to measure the height of objects, etc. Valuable suggestions along this line may be found in the Stone-Millis Plane Geometry, 1916 edition, and in other modern texts. It must be remembered that the purpose of this work is chiefly to stimulate interest. It can be overdone. It will be hard, for example, to persuade the intelligent pupil that he is really learning how to do the work of the world. In most cases the student, knowing of the instruments of precision used in making measurements, will soon grow weary of using his own crude instruments, and it is not always advisable to provide expensive instruments, this for obvious reasons. Again, it is not easy for the average teacher to keep a large group interested in a problem out-of-doors, even though a detailed report be required. Anything more than the occasional field trip for work out-of-doors is a waste of time and effort.

Interest may be further sustained by referring to the number of proofs for the Theorem of Pythagoras, especially to one

by President Garfield, and others recently discovered by high school students. Some of these may be found in back numbers of *School Science and Mathematics,* and in other periodicals.

The pupils will be interested in the three famous problems of antiquity, and they will be eager to go to encyclopedias and other sources for information about these problems. Some member of the class will be interested in constructing an instrument for trisecting an angle. Newell and Harper's Geometry may be used by the pupil for suggestions. They will be interested in seeing the value of Pi given to 707 places as it is in *Mathematical Wrinkles,* by S. I. Jones.

The attention of the pupils may be called to some of the great problems of geometry and of mathematics. For example: the discovery of the planet Neptune by mathematics; the location of big guns in the recent war; calculating distances of bodies remote from the earth; all of these to show the pupils how important geometry has been in the progress of civilization. Take them to see great bridges, buildings, great engineering achievements within reach, and show the important part mathematics had in their construction. The list of interesting things related to the study of geometry could be made endless. Those named are representative of what the alert teacher, who has a passion for his subject, may do. Under such guidance some pupils will be inspired with a lofe for the study of mathematics and determine to study the subject further because of the service it will enable them to render their fellows. They will be impressed with the fact that the world is only on the threshold of progress and that going further depends in large measure on the use of mathematics. Under proper instruction pupils will not lack motive for the study of mathematics.

The following are a few suggestions to be placed in the hands of the student or to be given orally to him by the teacher, on

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"How to Study a Theorem Which Is Proved in the Text."

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Their puprose is to aid the student in mastering the art of study, especially as applied to geometry. It would be an excellent idea to have such suggestions incorporated in the text to be used by the pupil, or placed in the pupil's hands in the form of a hand book on how to study mathematics.

The pupil should first ask himself, "What are the purposes to be aimed at and accomplished as far as possible, in the study of this theorem?" There are three important purposes, namely:

1. To learn the best way of studying, and of thinking out a problem.

2. To be able to use the theorem in proving other principles in geometry.

3. To use the theorem in solving problems encountered in life.

Which of these three purposes is of greatest concern to you? Is not the first by far the most important? Keeping this in mind will enable you to guard against the formation of improper habits of study. The formation of correct habits in this study will be of great assistance to you in your other studies. This study has advantages over many other studies in cultivating your ability to concentrate your mind on a problem and to ignore distractions. You can also be sure of your reasoning because you can test your results.

*Memorizing the Proof.* Since the most important thing connected with the theorem is the manner in which you study it, do not attempt to memorize the proof given in the text. That is, do not learn the steps by heart, for such would be mental slavery. Rather get the proof by some more thoughtful method. Learn to think for yourself and become independent of the book.

*Attempt to Find a Proof of Your Own First.* Do not permit yourself to look at the figure or proof in the text until you have first attempted to find a proof of your own. The author probably proved the theorem in full because he thought it difficult for you. He got all the fun which such work gives. The real pleasure is in doing it yourself. You may sometimes fail to find the proof yourself. But even then, as you will readily see, you have an advantage in the fact that you are able to understand and appreciate the author's proof better by comparison with the steps in your own effort. It is always exceedingly interesting and valuable to see how someone else easily accomplished a thing—like working a puzzle, for example—which you had attempted time after time with failure. The important points of the author's proof which you failed to see are more thoroughly impressed on your mind.

In attempting to find a proof of your own before looking at that in the text, determine carefully the hypothesis and conclusion, and be sure that you understand positively the meanings of all the terms involved. How do you find the meaning of a term when you do not know it? If you clearly understand the theorem you should be able to picture it clearly in your mind.

Draw a figure of your own, using your own lettering. Make the figure general. That is, if the theorem speaks of *any* triangle, do not draw a *right* triangle, nor an *isosceles* triangle. For you might be led by the special figure into a mistake in reasoning. Make the drawing accurate, because an accurately drawn figure will often suggest the proof. If it is at all convenient, use compass and straightedge in drawing the figure. Thus, if the figure is supposed to contain a perpendicular to a line, actually construct the perpendicular with the ruler and compass.

Now in endeavoring to discover the proof, it is usually best to begin by considering the conclusion

1. What kind of relation does the conclusion involve? Proving lines equal? Proving lines parallel? Proving angles equal? Proving a proportion? If it is proving lines equal, recall the different ways in which this is done—by congruence of triangles, opposite sides of a parellelogram, etc.

2. Recall any previous theorems, corollaries, or definitions thus related to the conclusion, on which the proof may be made to depend. Continue until the required links for connecting the hypothesis and conclusion are found.

3. If necessary, try to draw auxiliary lines that will help in the proof.

4. If you can find no direct proof in this way, think whether or not the indirect, or analytic, method of proof may be used. That is, proceed step by step from the conclusion to the hypothesis, asking at each point ho wthis result can be justified.

*Studying the Proof in the Text.* If you succeed in discovering a proof of your own, then read that of the text and compare the two. If you fail to discover a proof, study the author's proof. Observe especially the critical points of the latter which you failed to think of in your own struggle. These

are your weaknesses, and attention to them in this particular connection will impress them thoroughly on your mind.

*Looking Up the References.* The text does not usually give the reasons in full for the steps, but gives references in the margin to sections where these reasons may be found. The purpose of this is not merely to save space, but to give you opportunities to think for yourself. Hence, in studying the author's proof, do not look up a reference unless you cannot recall it. Test your accuracy by looking up the reference after you have decided what it is. Do this with especial care if there is any doubt in your mind. If you are sure you know the reference do not take time to look it up.

In reciting the proof orally, or in writing it, you should state in full the reasons for all steps. Do not refer to them by number, as that would place the useless burden on the mind of memorizing the number attached to each reason.

*Getting the Main Points of the Proof.* In studying the proof of the book, try to see the big ideas involved. Is the proof direct or indirect? Is the main idea congruence of triangles? Is it proportion? Is it a principle in parallelograms? Is it a comparison of angles? Is it superposition of figures?

Determine the most important steps of the proof. Pick out the critical points, or the turning points, of the proof. If these steps are mastered the others become minor details which are easily filled in.

To what other theorems is the present theorem most closely related?

It is this organization of the proof—getting the big ideas, picking out the most important steps, or turning points of the proof, and finding the most important relationships of the theorem to other theorems—that should be striven for.

*Writing the Proof.* After the proof of the theorem has been grasped it should be carefully written out without reference to the book. Your accuracy may be tested, if there is any doubt, by comparison with that of the text. Nothing will fix the proof more securely in mind than writing it out.

*Applying the Theorem.* Your knowledge of the theorem is not complete until you can use it, either in proving other principles of geometry, or in solving problems in practical situations.

Hence, examine the exercises in the book, especially those following closely after the theorem, to determine the use to be made of the theorem. Usually a number of such exercises can easily be found. Try to find out how to use the theorem in solving these exercises.

Be on the alert to find opportunity to use the theorem in some practical situation in your own work. Think of problems outside of the text in which the theorem may have application.

*An Example.* Let us illustrate the method outlined above by applying it to the study of the following theorem taken from the 1916 edition of the Stone-Millis text, page 76:

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*The line segment which joins the middle points of two sides of a triangle is parallel to the third side and is equal to one-half of it.*

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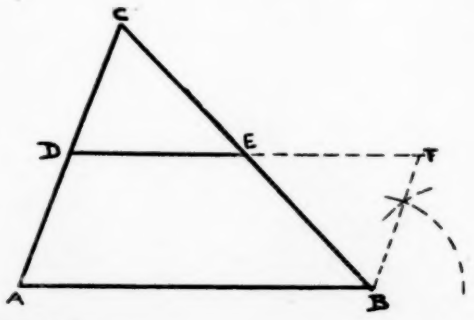
You will remember that the thing you value most is the learning how to study; hence, you will try to discover a proof for the theorem before looking at the one in the book.

Having first gotten the meaning of the theorem clearly in mind, so that you can imagine the figure, you will draw accurately with instruments the following figure to represent the theorem.




You will formulate:

(*a*) Hypothesis—In triangle $ABC$, $D$ is the middle point of $AC$, and $E$ the middle point of $BC$.

(*b*) Conclusion—$DE \parallel AB$, and $DE = \frac{1}{2}AB$.

In attacking the proof, you see that you are to prove lines parallel and line segments equal. There comes to your mind sev-

eral methods of accomplishing each of these. You weigh them: Lines are proved parallel by means of a transversal or by use of a parallelogram. Since you know nothing here about the angles formed by a transversal, you decide to use a parallelogram. Again, line-segments are possibly most often proved equal by congruence of triangles or by means of a parallelogram. The parallelogram method is chosen. Hence, in order to form a parallelogram, you produce $DE$ through $E$, and draw $BF \parallel AC$ to meet $DE$ at $F$.

You may be able to finish the proof or you may fail. In either case, you finally come to read the proof in the book which is as follows:

1. $D$ and $E$ are middle points of $AC$ and $BC$, respectively, *i. e.*, $AD = DC$ and $BE = EC$. Hyp.
2. Produce $DE$ through $E$, and draw $BF \parallel AC$, meeting $DE$ produced at $F$.
3. $\angle DEC = \angle BEF$. Verticle $\angle$s.
4. $\angle DCE = \angle EBF$. Sec. 29.
5. $\therefore$ Triangle $DEC$ is congruent with triangle $BEF$. Sec. 65.
6. $\therefore BF = DC$ and $EF = DE$. Def. of Congruence.
7. $\therefore BF = AD$. Ax. 1
8. $\therefore ABFD$ is a parallelogram. Sec. 90.
9. $\therefore DE \parallel AB$. Def. of Parallelogram.
10. Also $DF = AB$. Sec. 82.
11. $\therefore DE = \frac{1}{2} AB$. Ax. 5.

In each step try to think for yourself what theorem, corollary, or definition is the reason referred to. You should be able to recall it; but if not, then, and not before, is the time to look it up.

It is evident that you will be completely lost if you have not reviewed from day to day the theorems, etc., you have previously studied so that they may come to your mind when you need them.

Next get the big thought of the proof. What is it? Is it not the parallelogram? Is this the only idea of great importance? No, another is the congruence of triangles in proving $ABFD$ a parallelogram. Which, then, are the important steps of the proof? Steps 5 and 8. All other steps are of less importance, and relate themselves to these turning points of the

proof. Fix in mind, then, the movement of the proof, from the congruence of triangles to the parallelogram.

When you are sure that you thoroughly understand the proof write it out in full and then compare what you have written with the text.

Study the proof to find other ways of handling it. If you find another way be prepared to offer this in class. This will not only give you pleasure and satisfaction, and improve your standing in the class; but it will be an incentive to others to do the same sort of work. For example, in this theorem we might have produced $DE$ to $F$ making $EF = DE$, and have drawn $FB$. The triangles could then be proved congruent because two sides and the included angle of one would be equal respectively to the two sides and the included angle of the other.

Look among the theorems, corollaries, and exercises that follow this theorem to find some to which it applies. You will find the following:

The line-segment connecting the middle points of the non-parallel sides of a trapezoid is parallel to the bases and equal to one-half the sum of the bases.

The line-segments connecting the middle points of the adjacent sides of any quadrilateral form a parallelogram.

The line-segments connecting the middle points of the opposite sides of any quadrilateral bisect each other.

If $ABCD$ is a parallelogram, and $E$ and $F$ are the middle points of $DC$ and $AB$, respectively; prove that $AE$ and $CF$ bisect $DB$.

The medians of a triangle are concurrent at a point of trisection of each (two proofs).

It will also help us to establish the fact that the middle point of the hypotenuse of a right triangle is equally distant from the three vertices; and that if the hypotenuse is double the shorter leg the acute angls are 30° and 60°.

Try to prove these exercises, and others you may be able to find. It may be necessary to wait until other facts are proved before you can prove some of these exercises. Make a note of such and be on the look out for the needed information as the work proceeds.

What is the converse of this theorem? Do you find any proof for this?

Determine the relative value of this theorem as compared with other theorems, and be prepared to give reasons in the class for your decision. Compare your classification of the theorem with that of other members of the class. Note carefully any suggestions the teacher may make in this connection.

Keep a note book and record all the interesting constructions not given in the text, which have come up in the work. Keep a careful record of all facts, sidelights, and applications of the theorems proved, and indicate carefully the place in the text to which these apply.

If the spirit of these directions is followed, as well as the letter, any student can master geometry and obtain from its study the benefits to be derived from it.

To summarize briefly, as a student you should:

1. Review what has preceded so as to insure its recall when needed.
2. Read the statement of the theorem.
3. Without reference to the proof in the text try to work out a proof of your own, remembering that you have in hand all the information needed for the proof.
4. Compare your proof with that of the book.
5. If you do not succeed, or if you have made a mistake, master the points in which you have failed, with the determination to succeed the next time.
6. Be sure to identify the main, or critical, points of the proof. When these are mastered the rest is easy.
7. Try to recall all references to reasons. If you are not sure look them up again making an effort to remember them permanently.
8. It is usually advisable to write out the theorem with the text closed, and then compare what you have written with the text.
9. Relate the theorem to facts already studied, and apply it to exercises.

*(To be continued)*

## NEWS AND NOTES

W. D. REEVE, formerly head of the department of mathematics in the University of Minnesota High School, has been elected to the principalship of the school. Mr. Reeve will continue to give the professional courses in the School of Education on the teaching of secondary mathematics.

DR. FRANK C. TOUTON, formerly state supervisor of high schools in Wisconsin, is now lecturer of secondary education in the University of California.

THE September, 1921, issue of *Chicago Schools Journal* contains the new Chicago course of study in mathematics for the seventh and eighth grades. It is the work of the Course of Study Committee of the Chicago Principals' Club, assisted by a number of district superintendents, teachers selected from these grades, the director of school research, and the members of the department of mathmatics in the Chicago Normal College. No more complete and modern statement of a course of study in the mathematics of these grades has appeared.

FRANCES A. RODDY of Hackettstown, N. J., has invented an apparatus for illustrating and demonstrating the principles which underlie the composition and divisibility of numbers, or the principles of pure arithmetic.

THE *Newark Evening News* of Newark, N. J., under date of October 4, 1921, comments editorially regarding the policy recently adopted by the educational authorities of Ohio in respect in secondary mathematics:

MAKING MATHEMATICS A HIGH SCHOOL ELECTIVE

Ohio has eliminated mathematics as a compulsory study in its high schools. The effect probably will be to improve the "tone" of the average mathematics class. The good mathematicians will elect the subject and shine in it, and those not particularly strong in the reasoning processes will escape the rigorous mental calisthenics it entails, although they are the very ones that need it the most.

This is conceding a good deal to the 'non-mathematical mind,' a phrase which is too often a mere label for a flabby, lazy mind. The worth of mathematics as a mental discipline is too firmly established to require argument at this late day. To relegate it to the position of a mere elective in the secondary schools signifies too great a willingness to look upn the mind as a mere container of what is poured into it, rather than as a going machine, whose powers increase by exercise. The development of the logical faculty is of supreme importance to those who would interpret clearly the life of which they are a part. The despised algebra and the mysteries of geometry—however much they may appear, through uninspired teaching, to be blind alley studies, leading nowhere in a workday world—are keys that open the door to such a development, and are not lightly to be tossed away .

Anything that savors of a free elective system has no place below the college or university, and there—even in Harvard, which, next to the University of Virginia, has been most noted for its emancipation of the undergraduates' choice—the trend of late has been in the opposite direction. In seeking to avoid any arbitrarily standardized education that permits of no flexibility, shall we not do well to continue to realize that there is an indispensable minimum, say, of English, history, mathematics and perhaps of natural science, of which every American high school diploma should be a guaranty.

Our forty-eight state educational systems continue, fortunately, to be laboratories to test out theories borne in on any educational breeze. The Ohio experiment will be watched with deserved interest, and no serious objection can be had to it—by those who do not happen to live there.

THE program of the Central Association of Science and Mathematics Teachers given in St. Louis, November 25th and 26th, consisted of:

Methods in Beginning Classes in Plane Geometry, A. J. Schwartz, Cleveland High School, St. Louis, Mo.

What Shall Constitute the Material of the Mathematics Courses of the Seventh and Eighth Grades? Report of a Committee of Junior and Senior High School Teachers, presented

by M. J. Newell, Evanston Township High School, Evanston, Ill.

Problems Involved in the Scientific Construction of the Mathematics Curriculum for Grades Seven, Eight and Ninth, Raleigh Schorling, Lincoln School, New York.

The Project Method in Mathematics, Byron Cosby, State Teachers College, Kirksville, Mo.

The Teaching of Arithmetic in the High School, L. Gilbert Dake, Soldan High School, St. Louis, Mo.

Work Versus Marks, Murray A. Dalman, Director, Department of Reference and Research, Indianapolis, Ind.

Financial Report of the National Committee on Mathematical Requirements, Eula Weeks, member of the committee, Cleveland High School, St. Louis, Mo.

The officers of the Mathematics Section are: W. E. Beck, Chairman; Alfred Davis, Vice Chairman, and Elsie G. Parker, Secretary.

THE program of the Mathematics Section of the University of Illinois Conference, at Urbana, November 18, 1921, included:

"Fundamental Principles of Elementary Algebra," R. L. Modesitt, Charleston. Discussion, W. J. Risley, Decatur.

"Teaching Algebra without Home Work," H. C. Wright, Chicago. Discussion, Florence Morgan, Highland Park.

"The Proper Amount of Drill in Teaching Elementary Algebra," Emma C. Ackerman, Lockport. Discussion, E. L. Mayo, Joliet.

"Is There Any Relation Between General Intelligence and Ability to Do High School Algebra?" E. W. Schreiber, Maywood, and C. M. Austin, Oak Park.

"Intuitive Geometry. What? Where Taught?" G. A. Harper, Kenilworth. Discussion, Mabel Sykes, Chicago.

Symposium on Report of National Committee on Mathematical Requirements—The Reorganization of Mathematic Courses in Secondary Schools: J. T. Johnson, Chicago; Jessie D. Brakensick, Quincy; Georgia Fischer, St. Charles; Grace Madden, Champaign; J. K. McDonald, Decatur; Ruth Utley, Rockford.

"An Experiment in Teaching Mathematics in Several Central Illinois High Schools," Lida C. Martin, Decatur. W. T. Felts of Carbondale presided.

PROF W. H. METZLER has been elected Dean of the College of Liberal Arts at Syracuse University.

In the article entitled "Comments on the Teaching of Geometry" in the May issue of the MATHEMATICS TEACHER, the title and position of Mr. Touton is given as Professor of Secondary Education, University of California, Berkley. When Mr. Touton read proof on this article in July, it was signed: "Frank C. Touton, State Supervisor of High Schools, Madison, Wisconsin." The change was made by the editor without Mr. Touton's knowledge and should not have been made. Mr. Touton requests that this notation be inserted in the News and Notes section of the November issue, for his present position is that of Lecturer in Education, University of California, Berkley, California.

THE Board of Education of Newark, New Jersey, in April adopted a new schedule for teachers in high schools as follows: Teachers in junior high schools, $1,900 to $3,000; teachers in senior high schools, including librarians, $2,100 to $3,800; heads of departments, $2,700 to $4,400. The first—and last—named groups must serve ten years to reach the maximum, the second group twelve years. Teachers in the all-year high school, which is one of the unique features of the Newark schools, receive two months' extra pay, that is, 20 per cent. of the amount named. At the same time a statement of policy was adopted embodying several novel features in the form of joint conference committees to which are elected representatives of the teaching corps. One such committee has first authority over questions of adjustment of individual salaries; another over courses for teachers who desire recognition for graduate or other advanced study; while a third, the academic council, meets periodically with the Board of Education to consider specific questions of school policy. The last named has been in existence for several years with great success to its credit. Newark teachers are also granted leave of absence for study after ten years' service, with the loss of a fraction, less than half, of their salary. These features place Newark in the foremost of American cities in public educational policy.

*Should Text Books Provide Historical Notes?* I think historical material can be used to good advantage. They do not have to be the notes found in our text books, neither does such historical material need to come at the time when such notes occur in the text books. At the same time I find that some of the notes in our books are very timely. The students read many of them with interest.

From discussions in classes I would conclude that most students think of mathematics in some such terms as, "as it was in the beginning, is now, and ever shall be." Some have told me that they thought that mathematics had its day; that all we did was to keep it as nearly alive as we can. Discussions in the class will evoke correlations from the history, or story of mathematics, if you please. The teacher should be alert to take hold of such opportunities. Better than formal talks on the history, are short allusions made psychologically. I refer particularly to elementary mathematics classes.

J. Calvin Funk.

Santa Maria, Cal.

*Some Qualifications for Junior High School Teachers of Mathematics.* The teachers of mathematics in the junior high schools cannot be the grade teachers as such without additional training. Even less can such teachers be those who lack adaptibility and have received their training in a university only, where they have very likely acquired the methods of their professors. Such teachers, especially if thy have never taught before, are likely to be teachers of subjects, not of pupils. However necessary a good training is for successful teaching, we must nevertheless not forget that training does not necessarily assure us of ability to teach. While the work of the grade teacher is to a great extent drill work, and that of the high school teacher of a more specialized nature, the work of the junior high school teacher stands out as unique, especially that of the second and third year's work. The psychology of the situation is important. Instead of seeing principally the subject matter, the teacher should be able to see in his mind's eye the whole com-

munity before him, see the homes from which the pupils come, see th different occupations which the parents foilow, and see the different lines of work or vocations in which the pupils are interested. He should acquaint himself with at least some of the mathematics of these vocations. With this situation before him, he cannot help but make mathematics real and vital to the students. We frequently say that the teacher should, during this unique period, give the student a glimpse of what mathematics really is, give him a little touch of the different studies in high school mathematics. If the work is handled as I have outlined above and as I shall further comment, such a glimpse into more scientific mathematics will be quite natural.

High school teachers who have taught in the grades, ought to be promising candidates, for the work in the grades should have made them more sympathetic. It seems that teachers who have taught in the grades are more desirable than teachers who have taught only in high schools. I mean grade teachers who are students, who are eager for self-improvement, ever willing to widen their knowledge and broaden their horizon. Such teachers should have had four years of mathematics beyond the grades as a minimum. Taking for granted, that the head of the mathematics department understands the mission of the junior high school; that he has recently taught, or that he is teaching a class in the first year of high school mathematics, so that he sees clearly the weaknesses and problems in this year's work; he can, in my opinion, meet the emergency quite well until training schools train teachers for this work. If he has taught also in the upper grades so that these problems are not new to him, and if his training in education functions in his work, he is, in my opinion, ideally fitted to cope with the situation. He should give a short series or course of lectures to the prospective or entering teachers. Suggestive topics for such course would be, the significance and aim of junior high school mathematics, importance of the equation ,the formula, the relation of the equation to the graph, the importance of the graph with numerous illustrations, and others. When the teachers are thus filled with the subject, they can be of some inspiration to the pupils. It is needless to say that the head of the deparment should keep in close and sympathetic touch with the work as it progresses.

J. Calvin Funk.

## NEW BOOKS

**Greek Mathematics and Science.** By SIR THOMAS L. HEATH. Cambridge, the University Press, 1921, pp. ii+23.

This little monograph presents, in the dignified topography of the Cambridge University Press, a paper read before a joint meeting of the Classical Association (Leeds and District Branch), the Mathematical Association (Yorkshire Branch), and the Yorkshire Natural Science Association, at the University of Leeds, on March 5, 1921.

The American reader will probably be conscious of a feeling of surprise that three such associations should have joined to hear a paper that is devoted chiefly to Greek mathematics, even when presented by the world's leading authority upon the subject. That a classical group should join with a mathematical one might have been expected, even in America, but that an association devoted to natural science should also have joined in the meting would, unfortunately, be quite unheard of in this part of the world.

A second surprise that will come to the reader is due to the large amount of interesting and instructive material that has been condensed in so few pages—material with which a scholar will find himself somewhat familiar in most cases, but which will also contain much that is new to nearly everyone who opens the pamphlet. It is so fashionable in this country to dismiss Greek as a museum antiquity, and to devote the time that the language formerly consumed to such branches as natural science and sociology, that our teachers have lost the interest that their predecessors had in the greatest civilization of ancient times—a civilization to which the world is today so largely indebted.

And yet, in geometry particularly, a subject essentially Greek, the teacher is greatly handicapped who does not know the etymology of the terms in common use. Moreover, it is very suggestive in relation to methods of arousing an interest in the subject to know what "equilateral" means and that it is only a Latin translation of *isopleuros*, which contains the *iso* of "isosceles" and "isothermal," and the *pleuros* of "pleurisy." What a new interest comes into the rather dull vocabulary of the

science when we see the relation of "rhombus" to a spinning top, and when we know that the Greeks called an angle by the suggestive word *glochis* (arrow head), that a surface was to them a *chroia* (color, or skin), and that a point was a *stigme* (puncture)! The Greek terms are, of course, of little moment, but the naive way of naming the geometric concepts will add a good deal of interest in teaching the subject.

There is, however, much more than this in the monograph. There is, for example, mention of the prophetic vision of the Greek geometers with respect to astronomy, and of the essential features of the doctrines of Pythagoras. There is also a statement, well worth reading, setting forth the influence of the Pythagorean brotherhood, even after the death of the founder. And with all this there is a brief discussion of the indebtedness of science to the labors of men like Aristotle, Heraclides, Democritus, and Anaxagoras.

The pamphlet can, no doubt, be ordered through the New York correspondent of the Cambridge Press, and is well worth reading by those who are interested in the better teaching of mathematics.

DAVID EUGENE SMITH.

**Èléments de Géométrie.** By ALEXIS-CLAUDE CLAIRAUT. Paris, Gathier-Villars et Cie, 1920, 2 vols. Vol. I, pp. xiv + 95; vol. II, pp. 103.

Probably the most trite remark that anyone of our profession can make is that education is at present undergoing a tremendous change. The remark is true—that is, it is more or less true; it is true to us, just as a tree in a forest may seem gigantic to our eyes, but a commonplace thing to the aviator a mile above the earth's surface. Nevertheless it is probably a necessity that what we do should seem to us big with possibilities, so that we may be spurred on by laudable ambition.

Those of us who wish to see elementary geometry placed upon a more satisfactory foundation, at least so far as the teaching of the subject goes, will probably agree in a general way to such propositions as the following: (1) Geometry should begin with intuitive work, and especially with measuring; (2) in particular, we should first lead our pupils to employ the most natural means of measuring distances out of doors; (3) still

more particularly, the pupils should see the value of indirect measurement in such cases as that of finding the distance across a stream; (4) simple constructions (drawing of perpendiculars, bisecting line segments, and the like) should early form a "bond" with motor activities; (5) figures should be studied as needed, not necessarily in "water-tight compartments"; (6) the incommensurable should not trouble the pupil in the initial stages; (7) the intuitive work in mensuration should lead to "something worth while," like the finding the area of an irregular field; (8) congruence propositions may well come rather late in geometry, instead of being given first as with Euclid; (9) simple instruments, such as the pupil can easily make, should play an important part; (10) deductive work should develop gradually from intuitive geometry; (11) similarity should lead to the drawing of figures to scale and to the finding of distances and areas by the use of the resulting drawings; (12) the introduction to demonstrative work may well dispense with the extreme formalism of Euclid, and with any extended array of axioms; (13) the pupil should be allowed to postulate many geometric facts that Euclid would have proved; (14) the division of the subject matter into "books" is not a necessity, however much it may be desirable for certain reasons; (15) there is more than one method of demonstrating a theorem like the Pythagorean Proposition, and it is desirable to experiment upon others besides the one that Schopenhauer called "the rat-trap proof"; (16) it is not necessary to take as many propositions as are found in the older books, and the corollary is coming to be considered much less valuable than was once thought; (17) solid geometry may also be much reduced in extent and much simplified in presentation; (18) the essential feature of good teaching is to make the subject interesting.

All these points are recognized by M. Clairaut; they have the flavor of the best experimental schools of today; and any one of them might well be the subject of an earnest address before an association of teachers of mathematics. For this reason, we may well welcome this recent publication and may commend it to all who wish for better geometry in our high schools.

Another point of value in relation to the little treatise is that it is the work of one of the best known mathematicians of

France. To have a thoroughly modern presentation of geometry, embodying the best thoughts of some of our most advanced teachers, and presented by such a genius as M. Clairaut—this is refreshing.

What is the most interesting thing about the work, however, is that Clairaut, one of the world's infant prodigies, wrote it in 1741. When only ten years of age he had read Lhopital's analytic treatise on conics; when only thirteen he presented to the Academie des Sciences a memoir of higher curves; when he was sixteen years of age he presented another memoir on curves of double curvature; two years later he was admitted to the Academie; when he was twenty-three he was made a member of a committee composed of some of Europe's greatest scientists in the measurement of a degree of a meridian; and when, at the age of fifty-two, he passed away, France lost one of her most brilliant writers and imaginative mathematicians.

Perhaps the greatest lesson that this reprint of Clairaut's well-known geometry can teach us is the lesson of humility. We feel that we have made some great discovery in education, that some committee of which we are a member is revolutionizing the curriculum, that our pet experiment is the most important one that has ever been made in that field, that the past is mostly bad but that we shall help to make the future mostly good. Such feelings are natural; the world's inventions and discoveries have been made as the result of such confidence in self. But it is well for us all to understand that our ideas have rarely any element of novelty and that some portion of our time may profitably be spent upon a study of the attempts of our predecessors.

In mathematics we have always had the contest between the utilitarian and the ideal; between the "real problem" and the fanciful; between the psychological and the logical in the presentation of the subject. The "problem method," the "project," the "laboratory method," "mathematics in the field," the "informational problem," "general mathematics," "fusion," the "ratio method"—all these and such as these have been as thoroughly presented in the centuries past as Clairaut presents the intuitive and informal approach to geometry in this little classic. All of this should make us humble in our great educational theories, but it should not in the least discourage

us in the attempt to advance. Rather should it inspire us to read with greater care the works of the leaders in the world's progress and to profit by the permanent features which they contain.

DAVID EUGENE SMITH.

**Junior High School Mathematics.** By WALTER W. HART. D. C. Heath and Co., New York, 1921. Pp. 226.

The first of a three-book series, designed for the seventh grade. Emphasizes the fundamentals, intuitive geometry, and percentage.

**The Alexander-Dewey Arithmetics.** By GEORGIA ALEXANDER and JOHN DEWEY. Longmans, Green and Co., New York and Chicago, 1921.

A three-book series of arithmetics for the first eight grades. More than usual emphasis upon socialization and informational backgrounds of arithmetic.

**Mathematics for Shop and Drawing Students.** By H. M. KEAL and C. J. LEONARD. John Wiley and Sons, New York, 1921. Pp. 213.

For industrial workers and students who have not completed the courses in high school mathematics. Develops the essentials of mathematics for these students through illustrations from the shop and laboratory.

**Mathematics for Electrical Students.** By H. M. KEAL and C. J. LEONARD. John Wiley and Sons, New York, 1921. Pp. 230.

Similar to above, with illustrations from the field of electricity.

**Arithmetics.** By EUGENE HERZ and MARY G. BRANTS. The John C. Winston Company, Philadelphia, 1920.

A three-book series for the elementary school, emphasizing commercial applications.

**The Anderson Arithmetics.** By ROBERT F. ANDERSON. Silver Burdette and Co., Boston, New York and Chicago, 1921.

In these three books the author claims to have used "the results of modern school practice, experiments, and investigations contributed by many who have labored in this field to discover the inherent difficulties in the subject itself, to improve methods of instruction, and to eliminate useless subject matter."

**Analytic Geometry.** By CLAUDE IRWIN PALMER and WILLIAM CHARLES KRATHWOHL. McGraw-Hill Book Company, New York, 1921. Pp. 347.

Treats plane and solid analytic geometry and gives a brief introduction to the Calculus.

**Arithmetical Essentials.** By J. ANDREW DRUSHEL, MARGARET E. NOONAN, and JOHN W. WITHERS. Lyon and Carnahan, Chicago, 1921.

A three-book series for the elementary school. The treatment of Practice Tests alone demonstrates that this is not just another series of arithmetics.

**Games and Play for School Morale.** By MEL SHEPPARD and ANNA VAUGHAN. Published by The Community Service, Madison Ave., New York.

It is always difficult for a layman to decide just what games are best suited to children at their various ages. The problem is, of course, to get something that will be interesting enough to capture the attention of the child, and which will bring into play the characteristics peculiar to the particular period of life in which he is.

"Games and Play for School Morale" will appeal to all who have charge of the recreation hours of children, whether in the schoolroom or on the playground. The games are graded—from purely imaginative ones for small children, to volley ball, Hindu tag, and Indian club wrestling for the eighth grade groups. All are simply and concisely explained.

The last section is given over to group games for adults. Anyone who has ever had charge of a school or community social realizes that it is not an easy task to find sufficient games to fill an evening with simple and wholesome entertainment conducive to sociability. The thirty games described will prove a boon to harassed club hostesses and school workers.